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A Note on Torsionfree Modules

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In this note, all rings considered will be assumed commutative, Noetherian, with identity elements and modules will be unitary and finitely generated. The terms codimension, dimension, homological dimension, etc. will have the same meaning as in [*1*].

We shall prove the following results over a regular local ring R:

(a) If M, N are R-modules such that $M \otimes M$ and $N \otimes N$ are torsionfree, then $M \otimes N$ is torsionfree.

(b) If $M \otimes M \otimes M$ and $N \otimes N$ are torsionfree then $M \otimes N$ is reflexive.

In [*4*], the following result has been established, using *M.* Auslander's result that a torsionfree module over an unramified regular local ring is rigid [*1*, *2.2.*]:

PROPOSITION 0. *Let R be an analytically irreducible local Cohen–Macaulay ring and M, N R-modules.*

(a) *If $M \otimes N$ is torsionfree, M and N are torsionfree*;

(b) *If, in addition, M, N, and $M \otimes N$ have finite homological dimensions, then*

$$\text{hd } M + \text{hd } N = \text{hd}(M \otimes N).$$

We recall

DEFINITIONS. Let R be a local ring.

(i) An R-module M is rigid if, for every R-module N,

$$\mathrm{Tor}_i^R(M, N) = 0 \qquad \text{implies} \qquad \mathrm{Tor}_j^R(M, N) = 0$$

for all $j \geqslant i$ ($i \geqslant 0$).

(ii) R is analytically irreducible if its completion is an integral domain.

We start with another proof of the following result:

PROPOSITION 1. (Auslander [1]–Lichtenbaum [2]). *Let (R, J) be a regular local ring and M, N R-modules such that $M \otimes N$ is torsionfree. Then*

(i) *M, N are torsionfree;*

(ii) $\operatorname{Tor}_i^R(M, N) = 0$ *for all $i > 0$;*

(iii) $\operatorname{hd} M + \operatorname{hd} N = \operatorname{hd}(M \otimes N)$.

Proof. Parts (i) and (iii) follow from Proposition 0. We show (ii) by induction on $\dim. R = n$.

For $n \leq 2$, since $\operatorname{hd} M + \operatorname{hd} N = \operatorname{hd}(M \otimes N)$ by (iii), we have $\operatorname{hd} M + \operatorname{hd} N < 2$.

So M or N is free and $\operatorname{Tor}_i^R(M, N) = 0$ for all $i > 0$.

Let $n > 2$. By inductive hypothesis, for every prime ideal P of R which is not maximal, $\operatorname{Tor}_i^{R_p}(M_p, N_p) = 0$ for all $i > 0$.

Suppose $\operatorname{Tor}_K^R(M, N)$ is not zero for some $K > 0$. Let l be the largest such K. Then

$$\operatorname{Supp}(\operatorname{Tor}_l^R(M, N)) = \{J\}$$

which implies

$$\operatorname{Ass}(\operatorname{Tor}_l^R(M, N)) = \{J\}.$$

Hence $\operatorname{codim}(\operatorname{Tor}_l^R(M, N)) = 0$. We then have

$$\operatorname{Codim} N = \operatorname{hd} M - l \quad [1, 1.2.]$$

By (iii), $\operatorname{hd} N + \operatorname{codim} N = \operatorname{hd} M + \operatorname{hd} N - l = \operatorname{hd}(M \otimes N) - l$. Since $\operatorname{hd} N + \operatorname{codim} N = n$ and $\operatorname{hd}(M \otimes N) < n$, we have $l < 0$. This contradiction shows that

$$\operatorname{Tor}_i^R(M, N) = 0 \qquad \text{for all} \quad i > 0.$$

Let R be a regular local ring and M, N R-modules which are not free. If $M \otimes M$ and $M \otimes N$ are torsionfree, $N \otimes N$ is not necessarily torsionfree. However, we shall prove

THEOREM 2. *Let R be a regular local ring and M, N R-modules. If $M \otimes M$ and $N \otimes N$ are torsionfree, then $M \otimes N$ is torsionfree.*

For any modules M, N if $M \otimes N$ is torsionfree, then $\operatorname{Tor}_i^R(M, N) = 0$ for all $i > 0$, (Proposition 1). Hence Theorem 2 will follow from:

PROPOSITION 3. *Let (R, J) be a local ring and M, N R-modules of finite homological dimensions. Suppose that*

$$\operatorname{Tor}_i^R(M, M) = 0 = \operatorname{Tor}_i^R(N, N) \qquad \textit{for all} \quad > 0.$$

Then

(a) $\mathrm{Tor}_i^R(M, N) = 0$ *for all* $i > 0$.

If R *is a domain, and either* $M \otimes M$ *or* $N \otimes N$ *is torsionfree, then*

(b) $M \otimes N$ *is torsionfree.*

Proof. Let $\dim R = n$ and $\operatorname{codim} R = s$. We prove both parts by induction on $\dim R = n$.

(a) For $n \leqslant 1$,

$$2 \operatorname{hd} M = \operatorname{hd}(M \otimes M) \geqslant 1$$

and

$$2 \operatorname{hd} N = \operatorname{hd}(N \otimes N) \leqslant 1 \quad [1, 1,3],$$

So M and N are free which imply $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 0$.

Let $n \geqslant 2$. For any non-maximal prime ideal P of R, by inductive hypothesis,

$$\mathrm{Tor}_i^{R_P}(M_P, N_P) = 0 \qquad \text{for all} \quad i > 0.$$

Suppose $\mathrm{Tor}_K^R(M, N)$ is not zero for some $K > 0$. Let l be the largest such K.

As in Proposition 1, we have

$$\operatorname{codim} N = \operatorname{hd} M - l.$$

Then $s = \operatorname{hd} N + \operatorname{codim} N = \operatorname{hd} M + \operatorname{hd} N - l$. Now

$$2 \operatorname{hd} M = \operatorname{hd}(M \otimes M) \leqslant s$$

and

$$2 \operatorname{hd} N = \operatorname{hd}(N \otimes N) \leqslant s.$$

Therefore $\operatorname{hd} M + l \leqslant \operatorname{hd} N \leqslant \operatorname{hd} M - l$ which implies $l \leqslant 0$.

This contradiction shows that $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 0$.

(b) If $n \leqslant 1$, we have seen in (a) that M and N are free. So $M \otimes N$ is free which implies $M \otimes N$ is torsion free.

Let $n \geqslant 2$. By inductive hypothesis, $(M \otimes N)_P$ is R_P-torsionfree for every nonmaximal prime ideal P of R. Therefore no nonzero nonmaximal prime ideal P belongs to $\mathrm{Ass}(M \otimes N)$.

Suppose $M \otimes N$ is not torsionfree.

Then $J \in \mathrm{Ass}(M \otimes N)$ which implies $\mathrm{codim}(M \otimes N) = 0$. By (a), $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 0$ and therefore

$$\mathrm{hd}\, M + \mathrm{hd}\, N = \mathrm{hd}(M \otimes N) = s. \tag{3.1}$$

Since $M \otimes M$ or $N \otimes N$ is torsionfree, either

$$2\,\mathrm{hd}\, M = \mathrm{hd}(M \otimes M) < s$$

or

$$2\,\mathrm{hd}\, N = \mathrm{hd}(N \otimes N) < s.$$

Then $2\,\mathrm{hd}\, M + 2\,\mathrm{hd}\, N < 2s = 2\,\mathrm{hd}\, M + 2\,\mathrm{hd}\, N$ by (3.1). Hence both M and N are free and therefore $M \otimes N$ is torsionfree which contradicts our assumption that it is not. We conclude that $M \otimes N$ is torsionfree.

We now have an application of Theorem 2.

As usual, $\mathrm{Hom}_R(M, R)$ will be denoted by M^*.

THEOREM 4. *Let (R, J) be a regular local ring and M, N R-modules such that $M \otimes M \otimes M$ and $N \otimes N$ are torsionfree.*

Then $M \otimes N$ is reflexive.

Proof. We use again induction on $\dim R = n$.

For $n \leqslant 2$,

$$3\,\mathrm{hd}\, M = \mathrm{hd}(M \otimes M \otimes M) < 2$$

and

$$2\,\mathrm{hd}\, N = \mathrm{hd}(N \otimes N) < 2.$$

So M and N are free and $M \otimes N$ is therefore reflexive.

Let $n > 2$. By inductive hypothesis, $(M \otimes N)_P$ is R_P-reflexive for every nonmaximal prime ideal P of R.

By Theorem 2, $E = M \otimes N$ is torsionfree and can therefore be embedded in E^{**}.

We suppose E is not reflexive and derive a contradiction.

$\mathrm{Supp}(E^{**}/E) = \{J\}$ which implies $\mathrm{codim}(E^{**}/E) = 0$. Hence

$$\mathrm{hd}(E^{**}/E) = n.$$

Now $\mathrm{codim}\, E^{**} \geqslant 2$ since $\mathrm{codim}\, R > 2$. Therefore $\mathrm{hd}\, E^{**} \leqslant n - 2$ and we have

$$\mathrm{hd}\, E = \mathrm{hd}(E^{**}/E) - 1 = n - 1.$$

Since

$$3 \operatorname{hd} M = \operatorname{hd}(M \otimes M \otimes M) \leqslant n - 1$$

and

$$2 \operatorname{hd} N = \operatorname{hd}(N \otimes N) \leqslant n - 1,$$

We have

$$3 \operatorname{hd} M \leqslant \operatorname{hd}(M \otimes N) \qquad \text{and} \qquad 2 \operatorname{hd} N \leqslant \operatorname{hd}(M \otimes N).$$

But $\operatorname{hd}(M \otimes N) = \operatorname{hd} M + \operatorname{hd} N$ (Proposition 1) and therefore $2 \operatorname{hd} M \leqslant \operatorname{hd} N \leqslant \operatorname{hd} M$ which implies M is free and consequently N is also free. Thus $M \otimes N$ is reflexive which is a contradiction.

COROLLARY 5. *Let R be a regular local ring. If M is a module such that $M \otimes M \otimes M$ and $M^* \otimes M^*$ are torsionfree, then M is free.*

Proof. Since M is torsionfree and R is a normal ring, M is free if, and only if, $M \otimes M^*$ is reflexive.

The corollary then follows from Theorem 4.

Remark. It is obvious that over a regular domain R, Theorems 2 and 4 hold while in Corollary 5, M will be projective.

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